Effect of finite boundaries on the Stokes resistance of an arbitrary particle

Part 2. Asymmetrical orientations

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A general treatment is given of the first-order effects of wall proximity on the increased resistance to translational motions of a rigid particle of arbitrary shape settling in the Stokes régime. The analysis generalizes a previous treatment (Brenner 1962) to the case where the principal axes of resistance of the particle may have any orientation relative to the principal axes of the bounding walls. It is shown that, to the first order in the ratio of particle-to-boundary dimensions, the increased resistance of the particle can be represented by a symmetric, second-rank tensor (dyadic) whose value is independent of particle shape and orientation.

1. Introduction

The motivation for the present work stems, in part, from the fact that the motion of an anisotropic particle settling through an unbounded fluid at small Reynolds numbers is not generally parallel to the gravity field (Brenner 1963); however, as is well known, the quantitative behaviour of particles settling in the Stokes régime is profoundly influenced by wall effects. To test experimentally the predicted behaviour of anisotropic bodies one must therefore know the appropriate wall corrections for the case of *asymmetric* particle motions relative to the container boundaries. The present paper furnishes a general treatment of this subject to the first order in c/l (c = characteristic particle dimension, l = characteristic boundary dimension) for particles and boundaries of any shape. A knowledge of the wall effects to this order usually suffices in most experimental investigations.

By replacing the 'walls' by a second particle, the techniques developed in the above context are also used to treat the first-order interaction of two particles of arbitrary shape settling through an otherwise unbounded fluid.

The central relation developed herein is a natural generalization of one given previously (Brenner 1962, hereafter referred to as Part 1) for symmetrical motions. In particular we shall show that if $\mathbf{F}$ be the hydrodynamic vector force on a rigid particle of any shape moving with vector velocity $\mathbf{U}$ through a bounded fluid, then, to the first order in c/l,

$$\mathbf{F} = -6\pi\mu c[\boldsymbol{\phi}_\infty^{-1} - \mathbf{k}c/l + o(c/l)]^{-1}.\mathbf{U}, \tag{1.1}$$

where μ is the viscosity and $\boldsymbol{\phi}_\infty$ and $\mathbf{k}$ are dimensionless, symmetric dyadics, which now appear in place of their scalar counterparts in Part 1. The superscript -1 denotes a reciprocal (inverse) dyadic. The characteristic dimensions c and l may be chosen arbitrarily as alterations in their definitions produce corresponding changes in the definitions of $\boldsymbol{\phi}_\infty$ and $\mathbf{k}$, respectively, in such a way that the overall result is unaffected. As in the case of its scalar counterpart in Part 1, the advantage of equation (1.1) resides in the fact that it separates the wall correction into distinct contributions from the particle ($\boldsymbol{\phi}_\infty$) and boundary ($\mathbf{k}$).

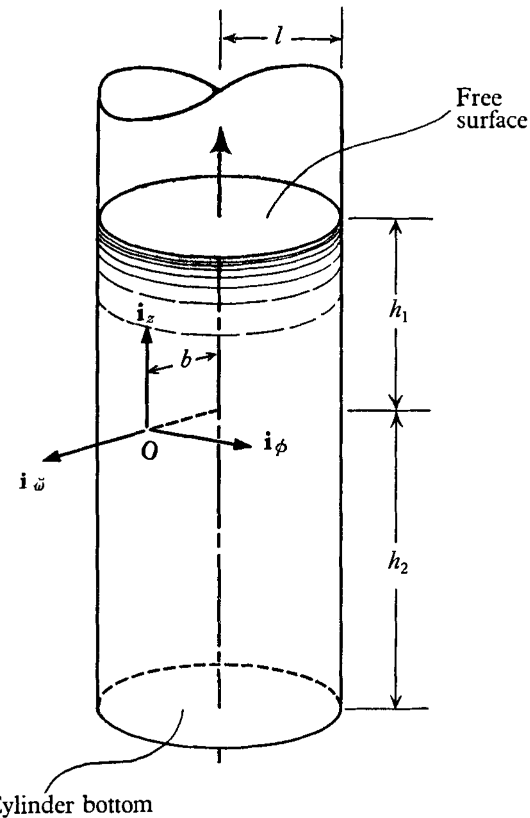

FIGURE 1. Principal axes of the wall-effect tensor in the interior of a circular cylinder.

The symmetric dyadic $\boldsymbol{\phi}_\infty$ is termed the Stokes resistance tensor of the particle (Brenner 1963). It is defined by the relation

$$\mathbf{F}_\infty = -6\pi\mu c \boldsymbol{\phi}_\infty . \mathbf{U}, \tag{1.2}$$

where $\mathbf{F}_\infty$ denotes the force which the body would experience if it moved through the *unbounded* fluid with velocity $\mathbf{U}$. It is an intrinsic and invariant property of the particle, dependent solely on the shape of the latter. In particular, $\boldsymbol{\phi}_\infty$ is independent of such factors as the size, velocity and orientation of the particle and of the properties of the fluid through which it moves.

We shall refer to $\mathbf{k}$ as the wall-effect tensor. It is an intrinsic property of the shape of the bounding walls and of the relative location of the 'centre' of the particle with respect to these boundaries. It is independent of such factors as the shape, size, orientation and velocity of the particle, the properties of the fluid, and size of the boundaries. As $\mathbf{k}$ is symmetric, it follows that at every point of a bounded fluid there exist a set of three mutually perpendicular axes such that if an isotropic particle, e.g. a sphere (whose centre coincides with this point) moves parallel to one of these axes, the force on the particle will be parallel to its direction of motion through the fluid.

To illustrate the general properties of this wall-effect tensor consider, for example, some point O within the interior of a circular cylinder of radius l filled to a finite depth with viscous fluid as in figure 1. The distance from the axis to O is b. The free surface and rigid cylinder base are located at distances h_1 and h_2, respectively, from O. If (ϖ, ϕ, z) are circular cylindrical co-ordinates having their origin at the axis and $(\mathbf{i}_\varpi, \mathbf{i}_\phi, \mathbf{i}_z)$ are the corresponding unit vectors at O, it is clear from symmetry considerations that the principal axes of $\mathbf{k}$ at O parallel to these vectors. Thus, we may write

$$\mathbf{k} = \mathbf{i}_\varpi \mathbf{i}_\varpi k_\varpi + \mathbf{i}_\phi \mathbf{i}_\phi k_\phi + \mathbf{i}_z \mathbf{i}_z k_z,$$

where the three scalars, k_ϖ, k_ϕ and k_z, are the principal values of $\mathbf{k}$ at O. They are each functions of the dimensionless distances b/l, h_1/l and h_2/l.

2. Formulation of the problem

Consider a rigid particle P settling with instantaneous velocity $\mathbf{U}$ in an otherwise quiescent fluid near some rigid boundary (or boundaries) S. The equations of motion and boundary conditions are:†

$$\nabla^2 \mathbf{v} = \mu^{-1} \nabla p, \tag{2.1}$$

$$\nabla . \mathbf{v} = 0, \tag{2.2}$$

$$\mathbf{v} = \mathbf{U} \quad \text{on} \quad P, \tag{2.3}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad S, \tag{2.4}$$

$$\mathbf{v} \to \mathbf{0} \quad \text{as} \quad r \to \infty. \tag{2.5}$$

One should, in general, allow in (2.3) for the possibility that the particle may also be rotating with some instantaneous angular velocity $\boldsymbol{\omega}$ as it translates through the fluid; however, as discussed in § 7, particle rotation does not affect the validity of (1.1) to the first order in c/l.

In Stokes flow the hydrodynamic force $\mathbf{F}$ exerted on a rigid particle moving with velocity $\mathbf{U}$ through a *bounded* fluid can be expressed in the general form

$$\mathbf{F} = -6\pi\mu c \boldsymbol{\phi} . \mathbf{U}, \tag{2.6}$$

where $\boldsymbol{\phi}$ is a (dimensionless) symmetric dyadic which is independent of the fluid properties and of the magnitude and direction of $\mathbf{U}$. It does, however, depend on all the other geometric aspects of the flow—namely, the sizes and shapes of P and S, the location of the centre of P relative to S and the orientation of

† Though unsymmetrical motions of the type under consideration are inherently unsteady, we have assumed that the *local* acceleration terms in the equations of motion may be neglected.

P relative to S. The proof of (2.6) and of the symmetry of $\boldsymbol{\phi}$ is substantially identical to that given for the analogous equation (1.2) for an unbounded fluid (Brenner 1963). The necessary modifications of the latter proof, arising from the presence of the boundaries S, are trivial since $\mathbf{v}$ vanishes identically on S.

The dependence of $\boldsymbol{\phi}$ on the geometrical configuration of the (P, S)-system is very complex in the general case. We now show, however, that to the first order in c/l, $\boldsymbol{\phi}$ may be resolved into separate contributions from P and S, as indicated in (1.1). As in Part 1, we utilize the 'method of reflexions'.

3. Method of reflexions

To solve the boundary-value problem posed by (2.1)–(2.5) we write

$$\mathbf{v} = \sum_{n=1}^{\infty} \mathbf{v}^{(n)}, \tag{3.1}$$

$$p = \sum_{n=1}^{\infty} p^{(n)}. \tag{3.2}$$

Each reflexion, $(\mathbf{v}^{(n)}, p^{(n)})$, is to satisfy the governing differential equations, (2.1) and (2.2). The boundary conditions (2.3)–(2.5) may be satisfied to any degree of approximation in c/l by successive application of the following boundary conditions to the individual reflexions:

$$\mathbf{v}^{(1)} = \mathbf{U} \quad \text{on} \quad P, \tag{3.3}$$

$$\mathbf{v}^{(2)} = -\mathbf{v}^{(1)} \quad \text{on} \quad S, \tag{3.4}$$

$$\mathbf{v}^{(3)} = -\mathbf{v}^{(2)} \quad \text{on} \quad P, \tag{3.5}$$

$$\mathbf{v}^{(4)} = -\mathbf{v}^{(3)} \quad \text{on} \quad S, \quad \text{etc.} \tag{3.6}$$

Also, for $n = 1, 2, 3, \ldots,$

$$\mathbf{v}^{(n)} \to \mathbf{0} \quad \text{as} \quad r \to \infty. \tag{3.7}$$

If $\mathbf{F}$ be the force exerted by the fluid on the particle, then (Part 1)

$$\mathbf{F} = \mathbf{F}^{(1)} + \mathbf{F}^{(3)} + \mathbf{F}^{(5)} + \ldots, \tag{3.8}$$

where $\mathbf{F}^{(n)}$ denotes the force on the particle associated with the nth reflexion.

The initial field $(\mathbf{v}^{(1)}, p^{(1)})$, satisfying the boundary conditions (3.3) and (3.7), corresponds to the motion of the particle through the *unbounded* fluid with velocity $\mathbf{U}$, the principal axes of the particle having the same orientation relative to $\mathbf{U}$ as in the bounded case. Hence, $\mathbf{F}^{(1)} = \mathbf{F}_\infty$ where $\mathbf{F}_\infty$ is given by (1.2). At large distances from any particle the asymptotic form of the initial field is (Part 1, (2.15)–(2.16))

$$\mathbf{v}^{(1)} = -\left(\mathbf{I} + \frac{\mathbf{rr}}{r^2}\right) \cdot \frac{\mathbf{F}_\infty}{8\pi\mu r} + o\left(\frac{c}{r}\right), \tag{3.9}$$

$$p^{(1)} = -\frac{\mathbf{r} \cdot \mathbf{F}_\infty}{4\pi r^3} + o\left(\frac{c}{r}\right), \tag{3.10}$$

where $\mathbf{I}$ is the idemfactor and $\mathbf{r}$ is the position vector of a point in the fluid relative to an origin at the 'centre' of the particle. The terms displayed explicitly in the above are the same as would arise in the unbounded fluid from the action of a *point force* of strength $\mathbf{F}_\infty$ situated at the origin.

Since $r = O(l)$ on S, it follows from (3.4) and (3.9) that the boundary condition to be satisfied by $\mathbf{v}^{(2)}$ on S is of the form

$$\mathbf{v}^{(2)} = \boldsymbol{\lambda}_S \cdot \frac{\mathbf{F}_\infty}{6\pi\mu l} + o\left(\frac{c}{l}\right) \quad \text{on} \quad S, \tag{3.11}$$

where
$$\boldsymbol{\lambda}_S\left(\frac{x_S}{l}, \frac{y_S}{l}, \frac{z_S}{l}\right) = \frac{3l}{4r_S}\left[\mathbf{I} + \frac{(\mathbf{r}_S/l)(\mathbf{r}_S/l)}{(r_S/l)^2}\right], \tag{3.12}$$

in which $\mathbf{r}_S \equiv (x_S, y_S, z_S)$ refers to a point on S. The dimensionless, variable dyadic $\boldsymbol{\lambda}_S$ is of $O(1)$ with respect to the parameter c/l. It is clear that, at any point on S, $\boldsymbol{\lambda}_S$ depends only upon the location of the centre of P relative to S and upon the geometrical shape of the latter. It is independent of the size of S.

In Stokes flow, $\mathbf{F}_\infty = O(6\pi\mu c\mathbf{U})$. Thus, the boundary conditions (3.7) and (3.11) uniquely define $\mathbf{v}^{(2)}$ to $O(c/l)$ for any given S. It follows that, to this order, $\mathbf{v}^{(2)}$ must be of the form

$$\mathbf{v}^{(2)} = \boldsymbol{\lambda} \cdot \frac{\mathbf{F}_\infty}{6\pi\mu l} + o\left(\frac{c}{l}\right), \tag{3.13}$$

where $\boldsymbol{\lambda} = \boldsymbol{\lambda}(x/l, y/l, z/l)$ is a dimensionless, variable dyadic of $O(1)$ which reduces to $\boldsymbol{\lambda}_S$ on S. Thus, at any point in the fluid, it too depends only upon the location of the centre of P relative to S and upon the shape of the latter.

Now (see (2.19), Part 1), the force $\mathbf{F}^{(3)}$ arising from $(\mathbf{v}^{(3)}, p^{(3)})$ is, in our present notation,

$$\mathbf{F}^{(3)} = 6\pi\mu c \boldsymbol{\phi}_\infty \cdot \mathbf{v}_P^{(2)} + o(c/l), \tag{3.14}$$

where $\mathbf{v}_P^{(2)}$ refers to the value of $\mathbf{v}^{(2)}$ at the centre of the space presently occupied by the particle. But, from (3.13), we have

$$\mathbf{v}_P^{(2)} = \mathbf{k} \cdot \frac{\mathbf{F}_\infty}{6\pi\mu l} + o\left(\frac{c}{l}\right), \tag{3.15}$$

where $\mathbf{k}$ denotes the value of $\boldsymbol{\lambda}$ at the centre of the particle, $(x = 0, y = 0, z = 0)$. $\mathbf{k}$ is therefore a dimensionless dyadic of $O(1)$, dependent only upon the relative location of the centre of P with respect to S and upon the shape of the latter. We shall show subsequently that $\mathbf{k}$ is symmetric.

Upon combining (3.14) and (3.15) we obtain

$$\mathbf{F}^{(3)} = \boldsymbol{\phi}_\infty \cdot \mathbf{k} \cdot \mathbf{F}_\infty (c/l) + o(c/l). \tag{3.16}$$

Though $\boldsymbol{\phi}_\infty$ and $\mathbf{k}$ are symmetric, their product is not generally symmetric and one must be careful to preserve the proper order of these dyadics in the multiplication.

Higher-order reflexions may be obtained as outlined in Part 1. One thereby obtains for $m = 0, 1, 2, \ldots$

$$\mathbf{F}^{(2m+1)} = (\boldsymbol{\phi}_\infty \cdot \mathbf{k} c/l)^m \cdot \mathbf{F}_\infty + o(c/l)^m. \tag{3.17}$$

If we now substitute (3.17) into (3.8) and sum the resulting geometric series, we obtain†

$$\mathbf{F} = [\mathbf{I} - \boldsymbol{\phi}_\infty \cdot \mathbf{k}(c/l) + o(c/l)]^{-1} \cdot \mathbf{F}_\infty. \tag{3.18}$$

† This is the analogue of the corresponding scalar relation

$$\frac{F}{F_\infty} = \left[1 - k\frac{|F_\infty|}{6\pi\mu|U|l} + o\left(\frac{c}{l}\right)\right]^{-1}$$

given in Part 1.

If $\mathbf{F}_\infty$ is eliminated from this expression via (1.2) we obtain equation (1.1), which constitutes our main result.

The proof that the $\mathbf{k}$ is symmetric now follows readily. From (1.1) and (2.6) we find that

$$\boldsymbol{\phi}^{-1} = \boldsymbol{\phi}_\infty^{-1} - \mathbf{k}(c/l) + o(c/l).$$

Thus, as $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_\infty$ are known to be symmetric the same must be true of $\mathbf{k}$.

Because of the appearance of the surface integral $\iint d\mathbf{S}.\mathbf{\Pi}.\mathbf{v}$ ($d\mathbf{S}$ = directed element of surface area parallel to the normal, $\mathbf{n}$, at the surface; $\mathbf{\Pi}$ = pressure tensor) in the reciprocal theorem employed in the derivation of (2.6) and (3.14), it follows that (1.1) is equally valid in the important case where one of the boundaries comprising S is a planar *free* surface, e.g. the upper surface of the liquid in a circular cylinder. For in this case the vanishing of the tangential stresses implies that $d\mathbf{S}.\mathbf{\Pi}$ is parallel to $\mathbf{n}$. This, in conjunction with the vanishing of the normal velocity, $\mathbf{n}.\mathbf{v}$, at the planar free surface causes the surface integral to vanish. One thereby formally obtains the same result as if the vector velocity $\mathbf{v}$ were itself zero on the surface, as heretofore assumed in the analysis. Since boundary conditions of the type (3.4), (3.6), etc., must be altered for a free surface, it is clear that the $\mathbf{k}$ value will be different for a free surface than a rigid surface.

4. Applications to a settling particle

For applications involving settling particles the hydrodynamic force $\mathbf{F}$ is known *a priori* (providing that we neglect the inertial force on the particle resulting from its linear acceleration) and its instantaneous velocity $\mathbf{U}$ is sought. Solving (1.1) explicitly for this velocity we obtain

$$\mathbf{U} = \mathbf{U}_\infty + \frac{\mathbf{k}.\mathbf{F}}{6\pi\mu l} + o\left(\frac{c}{l}\right), \tag{4.1}$$

where

$$\mathbf{U}_\infty = -\frac{\boldsymbol{\phi}_\infty^{-1}.\mathbf{F}}{6\pi\mu c} \tag{4.2}$$

denotes the velocity with which the particle would settle (for the same orientation) in an unbounded fluid. To compute $\mathbf{F}$ let $\mathbf{g}$ be the local acceleration of gravity vector, directed vertically downward, and let m_p and m_f, respectively, be the mass of the particle and displaced fluid. Thus (upon neglecting the inertial force on the particle mass) the gravitational, hydrostatic and hydrodynamic forces on the particle are in equilibrium. The constant hydrodynamic force on the particle is, therefore,

$$\mathbf{F} = -(m_p - m_f)\,\mathbf{g}. \tag{4.3}$$

Equations (4.1)–(4.3) show that even an isotropic particle will not generally settle vertically in a bounded fluid unless one of the principal axes of $\mathbf{k}$ lies parallel to the Earth's gravity field.

As a simple example of the application of (4.1) consider the motion of a thin, homogeneous, circular disk of thickness b and radius c ($c \gg b$) falling under the influence of gravity in a semi-infinite viscous fluid bounded below by an infinitely extended, rigid, plane wall as in figure 2. The instantaneous position of the centre of the disk from the wall is l. We let x_j, $\bar{x}_j$ and $\bar{\bar{x}}_j$ ($j = 1, 2, 3$) be co-ordinates fixed in space, fixed in the disk, and fixed in the wall, respectively. The x_1 co-ordinate is directed towards the centre of the Earth and the x_2 co-ordinate is parallel to the surface of the Earth. The '3' co-ordinates are all directed out of the plane of the paper. Unit vectors in the three different co-ordinate systems are denoted by $\mathbf{i}_j$, $\bar{\mathbf{i}}_j$ and $\bar{\bar{\mathbf{i}}}_j$. The angles made by the plane of the disk and wall, respectively, with the horizontal are denoted by ξ and η. We propose to calculate the components of the instantaneous settling velocity of the disk in the x_j system.

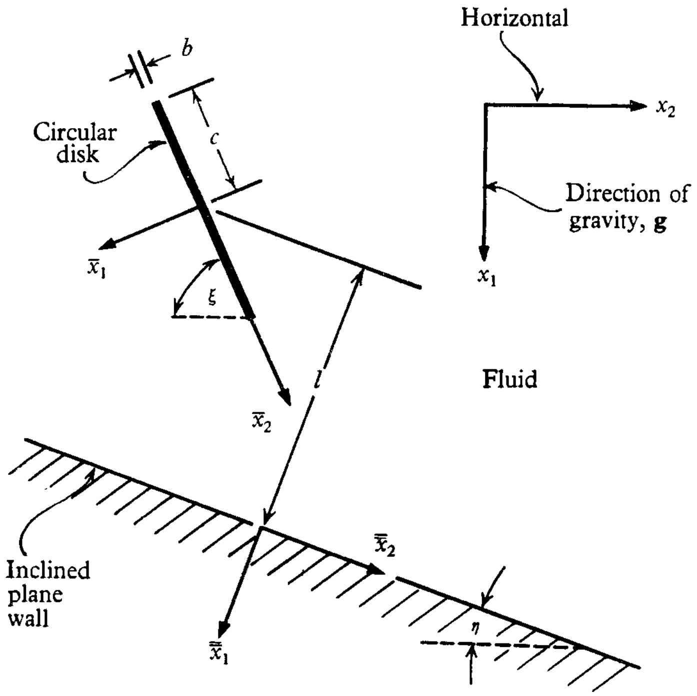

FIGURE 2. Circular disk settling asymmetrically near an inclined plane wall.

It is clear from symmetry that the principal axes of $\boldsymbol{\phi}_\infty$ lie parallel to the $\bar{x}_j$-axes. The Stokes force experienced by a circular disk moving broadside-on with velocity $\mathbf{U}$ in an unbounded medium is $\mathbf{F}_\infty = -16\mu c\mathbf{U}$ (Lamb 1932). The corresponding expression for edge-on motion is $\mathbf{F}_\infty = -(32/3)\mu c\mathbf{U}$. Hence, from (1.2), we obtain

$$\boldsymbol{\phi}_\infty = (8/9\pi)(\bar{\mathbf{i}}_1\bar{\mathbf{i}}_1 3 + \bar{\mathbf{i}}_2\bar{\mathbf{i}}_2 2 + \bar{\mathbf{i}}_3\bar{\mathbf{i}}_3 2). \tag{4.4}$$

It also follows from symmetry that the principal axes of $\mathbf{k}$ are everywhere parallel to the $\bar{\bar{x}}_j$-axes. Hence, from equations (3.4) and (3.6) of Part 1, we find that†

$$\mathbf{k} = \tfrac{9}{16}(\bar{\bar{\mathbf{i}}}_1\bar{\bar{\mathbf{i}}}_1 2 + \bar{\bar{\mathbf{i}}}_2\bar{\bar{\mathbf{i}}}_2 + \bar{\bar{\mathbf{i}}}_3\bar{\bar{\mathbf{i}}}_3). \tag{4.5}$$

† The comparable expression for a *free* plane surface is (see Part 1, (3.5) and (3.7))

$$\mathbf{k} = \tfrac{3}{8}(\bar{\bar{\mathbf{i}}}_1\bar{\bar{\mathbf{i}}}_1 2 - \bar{\bar{\mathbf{i}}}_2\bar{\bar{\mathbf{i}}}_2 - \bar{\bar{\mathbf{i}}}_3\bar{\bar{\mathbf{i}}}_3).$$

The plane surface, either free or rigid, appears to be the only boundary for which $\mathbf{k}$ has the same value at *all* points of the fluid.

We also note that $\mathbf{g} = \mathbf{i}_1 g$. Since $m_p - m_f = \pi c^2 b \Delta\rho$ ($\Delta\rho$ = difference in density of disk and fluid), a straightforward calculation yields $U_3 = 0$ and

$$\left.\begin{aligned} U_1 &= \frac{\pi c b g \Delta\rho}{32\mu}\left[(2+\sin^2\xi) - \frac{3}{\pi}(2-\sin^2\eta)\frac{c}{l} + o\left(\frac{c}{l}\right)\right], \\ U_2 &= \frac{\pi c b g \Delta\rho}{32\mu}\left[\sin\xi\cos\xi + \frac{3}{\pi}\sin\eta\cos\eta\left(\frac{c}{l}\right) + o\left(\frac{c}{l}\right)\right]. \end{aligned}\right\} \tag{4.6}$$

The only other boundary and position for which the wall-effect tensor is *completely* known is at the centre of a hollow sphere filled with viscous liquid. It is self-evident that $\mathbf{k}$ must be isotropic for this situation. Hence, if l is the radius of the hollow sphere, we find from equation (3.3) of Part 1 that

$$\mathbf{k} = \mathbf{I}\tfrac{9}{4}. \tag{4.7}$$

5. Moving boundary or net flow at infinity

Equation (1.1) is applicable only to the case where the boundary S is at rest and the fluid at infinity is at rest. It is of some interest to modify (1.1) so as to remove these restrictions. The case in which S is in motion arises, for example, when S is itself a particle falling in proximity to the original particle P. The case in which a net flow occurs at infinity arises, for example, during Poiseuille flow through a circular tube containing a particle. Problems of the latter type are of interest in connexion with the observed radial migration of particles in Poiseuille flow (Segré & Silberberg 1962; Goldsmith & Mason 1962).

In the event that either S moves or the fluid at infinity is in net flow, let $(\mathbf{v}^{(0)}, p^{(0)})$ denote the local velocity and pressure fields which would arise from either of these motions if P were absent from the fluid. It is assumed that these fields satisfy (2.1) and (2.2). To (3.1) and (3.2), respectively, we now add these initial fields $(\mathbf{v}^{(0)}, p^{(0)})$. Among the boundary conditions (3.3)–(3.7), only (3.3) requires modification. In its stead the proper boundary condition is now

$$\mathbf{v}^{(1)} = \mathbf{U} - \mathbf{v}^{(0)} \quad \text{on} \quad P. \tag{5.1}$$

The initial field $\mathbf{v}^{(0)}$ can be expanded in a Taylor series about the centre of P

$$\mathbf{v}^{(0)} = \mathbf{v}_P^{(0)} + o(c/l), \tag{5.2}$$

where $\mathbf{v}_P^{(0)}$ denotes the value of $\mathbf{v}^{(0)}$ at the centre of the space presently occupied by the particle. Thus, in place of (5.1), $\mathbf{v}^{(1)}$ is now uniquely defined to the first order in c/l by the relation

$$\mathbf{v}^{(1)} = \mathbf{U} - \mathbf{v}_P^{(0)} + o(c/l) \quad \text{on} \quad P. \tag{5.3}$$

Upon repeating the analysis which led to (1.1), we now find that the force on the particle is given correctly to the first order by the expression

$$\mathbf{F} = -6\pi\mu c[\boldsymbol{\phi}_\infty^{-1} - \mathbf{k}(c/l) + o(c/l)]^{-1} \cdot [\mathbf{U} - \mathbf{v}_P^{(0)} + o(c/l)], \tag{5.4}$$

where $\mathbf{k}$ has the same value as previously, i.e. in the case where $\mathbf{v}^{(0)}$ was identically zero.

Equation (5.4) in conjunction with (4.3) yields the instantaneous particle velocity

$$\mathbf{U} = \mathbf{v}_P^{(0)} + \frac{m_p - m_f}{6\pi\mu c}\left(\boldsymbol{\phi}_\infty^{-1} - \mathbf{k}\frac{c}{l}\right) \cdot \mathbf{g} + o\left(\frac{c}{l}\right). \tag{5.5}$$

This relation shows that a particle will undergo radial movement whenever no one of its three principal axes of resistance lies parallel to the Earth's gravitational field, whether the fluid is in Poiseuille flow† or is stagnant, and that, judging from the experiments of Goldsmith & Mason (1962), the shear distorts the particle so that its shape and orientation will allow such migration.

6. The motion of two particles through an unbounded fluid

Consider the motion of two particles (P_1 and P_2) of arbitrary shape settling with any relative orientations through an unbounded fluid. It is assumed that the characteristic particle dimensions (c_1 and c_2) are both small compared with their centre-to-centre spacing, l. The particles move with instantaneous velocities $\mathbf{U}_1$ and $\mathbf{U}_2$, respectively. The analysis which follows is a generalization of previous work on *spherical* particles (cf. Kynch 1959). The ability to bring the more general problem to fruition depends on the observation that the 'wall-effect' tensor, $\mathbf{k}$, which appears in (1.1) must somehow be related to the Stokes resistance tensor for the second particle and to the relative positions of the centres of P_1 and P_2.

For definiteness, we identify P_1 with the symbol P used earlier and P_2 with S. The boundary conditions are $\mathbf{v} = \mathbf{U}_j$ on P_j $(j = 1, 2)$ and $\mathbf{v} \to \mathbf{0}$ as $r_1, r_2 \to \infty$. Here, $\mathbf{r}_j$ denotes the position vector of a point relative to an origin at the centre of P_j. To solve this problem we utilize the modified reflexion scheme outlined in § 5 with $\mathbf{U}$ now replaced by $\mathbf{U}_1$. The initial field $(\mathbf{v}^{(0)}, p^{(0)})$ is to satisfy the boundary conditions

$$\mathbf{v}^{(0)} = \begin{cases} \mathbf{U}_2 & \text{on} \quad P_2, \\ \mathbf{0} & \text{at} \quad r_2 = \infty. \end{cases} \tag{6.1}$$

Then, from (5.4), the force on P_1 is, to at least $O(c_j/l)$,

$$\mathbf{F}_1 = -6\pi\mu c_1[(\boldsymbol{\phi}_\infty)_1^{-1} - \mathbf{k}_2(c_1/l)]^{-1}.(\mathbf{U}_1 - \mathbf{v}_{P_1}^{(0)}), \tag{6.2}$$

where $(\boldsymbol{\phi}_\infty)_j$ denotes the Stokes resistance tensor for P_j, and $\mathbf{k}_j$ is the 'wall-effect' tensor arising from the presence of the 'boundary' P_j in the fluid; $\mathbf{k}_j$ is a property only of the shape of P_j and of the relative location of the centre of the other particle with respect to the centre of P_j.

The asymptotic form of the initial field created by the motion of P_2 through the unbounded fluid in the absence of P_1 is, to $O(c_2/l)$,

$$\mathbf{v}^{(0)} = -\left(\mathbf{I} + \frac{\mathbf{r}_2\mathbf{r}_2}{r_2^2}\right).\frac{(\mathbf{F}_\infty)_2}{8\pi\mu r_2}, \tag{6.3}$$

where

$$(\mathbf{F}_\infty)_2 = -6\pi\mu c_2(\boldsymbol{\phi}_\infty)_2.\mathbf{U}_2 \tag{6.4}$$

denotes the force on P_2 in an infinite medium from which P_1 is absent. Now, let $\boldsymbol{\epsilon}$ be a unit vector drawn along the line of centres. It is immaterial whether it is directed from P_1 to P_2 or vice versa since $\boldsymbol{\epsilon}$ always appears in the dyad combination $\boldsymbol{\epsilon\epsilon}$. At the centre of P_1, $r_2 = \pm\boldsymbol{\epsilon}l$; hence, from (6.3) and (6.4),

$$\mathbf{v}_{P_1}^{(0)} = (3c_2/4l)(\mathbf{I} + \boldsymbol{\epsilon\epsilon}).(\boldsymbol{\phi}_\infty)_2.\mathbf{U}_2. \tag{6.5}$$

† Here, $\mathbf{v}_P^{(0)} = 2\mathbf{V}[1-(b/l)^2]$ where $\mathbf{V}$ is the mean velocity of flow through the tube, b = distance of centre of particle from cylinder axis, l = tube radius.

As indicated in § 5, to deduce the expression for $\mathbf{k}_2$ it is sufficient to consider the case where P_1 moves with velocity $\mathbf{U}_1$ while P_2 is at rest. Since $\mathbf{v}^{(1)}$ for this special case satisfies the conditions $\mathbf{v}^{(1)} = \mathbf{U}_1$ on P_1 and $\mathbf{v}^{(1)} \to \mathbf{0}$ as $r_1 \to \infty$, the asymptotic form of $\mathbf{v}^{(1)}$ is

$$\mathbf{v}^{(1)} = -\left(\mathbf{I}+\frac{\mathbf{r}_1\mathbf{r}_1}{r_1^2}\right).\frac{(\mathbf{F}_\infty)_1}{8\pi\mu r_1}. \tag{6.6}$$

At the centre of P_2 this has the value

$$\mathbf{v}^{(1)}_{P_2} = -(\mathbf{I}+\boldsymbol{\epsilon\epsilon}).(\mathbf{F}_\infty)_1/8\pi\mu l. \tag{6.7}$$

But, by a relation analogous to (3.14), the force on P_2 arising from the field $(\mathbf{v}^{(2)}, p^{(2)})$ (which satisfies the boundary condition $\mathbf{v}^{(2)} = -\mathbf{v}^{(1)}$ on P_2 and $\mathbf{v}^{(2)} \to \mathbf{0}$ as $r_2 \to \infty$) is

$$\mathbf{F}^{(2)}_2 = 6\pi\mu c_2(\boldsymbol{\phi}_\infty)_2.\mathbf{v}^{(1)}_{P_2}, \tag{6.8}$$

where $\mathbf{v}^{(1)}_{P_2}$ is given in (6.7). By virtue of this force the asymptotic form of $\mathbf{v}^{(2)}$ is

$$\mathbf{v}^{(2)} = -\left(\mathbf{I}+\frac{\mathbf{r}_2\mathbf{r}_2}{r_2^2}\right).\frac{\mathbf{F}^{(2)}_2}{8\pi\mu r_2}. \tag{6.9}$$

Thus, from (6.7) to (6.9), we find that the value of $\mathbf{v}^{(2)}$ at the centre of P_1 is

$$\mathbf{v}^{(2)}_{P_1} = \frac{9c_2}{16l}(I+\boldsymbol{\epsilon\epsilon}).(\boldsymbol{\phi}_\infty)_2.(\mathbf{I}+\boldsymbol{\epsilon\epsilon}).\frac{(\mathbf{F}_\infty)_1}{6\pi\mu l}. \tag{6.10}$$

Comparison with (3.15) shows that

$$\mathbf{k}_2 = \frac{9c_2}{16l}(\mathbf{I}+\boldsymbol{\epsilon\epsilon}).(\boldsymbol{\phi}_\infty)_2.(\mathbf{I}+\boldsymbol{\epsilon\epsilon}). \tag{6.11}$$

This dyadic is symmetric, in concordance with our general observation of the symmetry of $\mathbf{k}$.

Upon substituting (6.5) and (6.11) into (6.2) we obtain

$$\frac{\mathbf{F}_1}{6\pi\mu c_1} = -\left[(\boldsymbol{\phi}_\infty)_1^{-1}-\frac{9}{16}\frac{c_1}{l}\frac{c_2}{l}(\mathbf{I}+\boldsymbol{\epsilon\epsilon}).(\boldsymbol{\phi}_\infty)_2.(\mathbf{I}+\boldsymbol{\epsilon\epsilon})\right]^{-1}.\left[\mathbf{U}_1-\frac{3}{4}\frac{c_2}{l}(\mathbf{I}+\boldsymbol{\epsilon\epsilon}).(\boldsymbol{\phi}_\infty)_2.\mathbf{U}_2\right], \tag{6.12}$$

which constitutes the main result of this section. The corresponding force on P_2 may be obtained by permuting the indices.

In most applications of interest, $\mathbf{F}_1$ and $\mathbf{F}_2$ are given and one desires to calculate the settling velocities $\mathbf{U}_1$ and $\mathbf{U}_2$. This is readily done by simultaneously solving (6.12) and its counterpart for these velocities. As a simple example, let P_1 and P_2 be spheres of radii c_1 and c_2, respectively. The dimensionless resistance tensors are now isotropic and have the values $(\boldsymbol{\phi}_\infty)_1 = (\boldsymbol{\phi}_\infty)_2 = \mathbf{I}$. Substitution in (6.12) yields

$$-\frac{\mathbf{F}_1}{6\pi\mu c_1} = \left[\mathbf{I}-\frac{9}{16}\frac{c_1}{l}\frac{c_2}{l}(\mathbf{I}+3\boldsymbol{\epsilon\epsilon})\right]^{-1}.\left[\mathbf{U}_1-\frac{3}{4}\frac{c_2}{l}(\mathbf{I}+\boldsymbol{\epsilon\epsilon}).\mathbf{U}_2\right].$$

Now let $\mathbf{V}_1$ be the Stokes-law velocity with which P_1 would settle under the influence of gravity in an infinite medium from which P_2 was absent. Then $\mathbf{F}_1 = -6\pi\mu c_1\mathbf{V}_1$, which relation we employ to eliminate $\mathbf{F}_1$ from the above.

A second relation of similar form is then obtained by permuting the indices. Upon solving the two resulting equations simultaneously for $\mathbf{U}_1$ and $\mathbf{U}_2$ with the aid of the identity

$$\left[\mathbf{I}-\frac{9}{16}\frac{c_1}{l}\frac{c_2}{l}(\mathbf{I}+3\boldsymbol{\epsilon\epsilon})\right]^{-1} = (\mathbf{I}-\boldsymbol{\epsilon\epsilon})\left(1-\frac{9}{16}\frac{c_1}{l}\frac{c_2}{l}\right)^{-1}+\boldsymbol{\epsilon\epsilon}\left(1-\frac{9}{4}\frac{c_1}{l}\frac{c_2}{l}\right)^{-1}$$

we obtain, after considerable reduction, the expression

$$\mathbf{U}_1 = \mathbf{V}_1+\frac{3}{4}\frac{c_2}{l}(\mathbf{V}_2+\boldsymbol{\epsilon\epsilon}.\mathbf{V}_2),$$

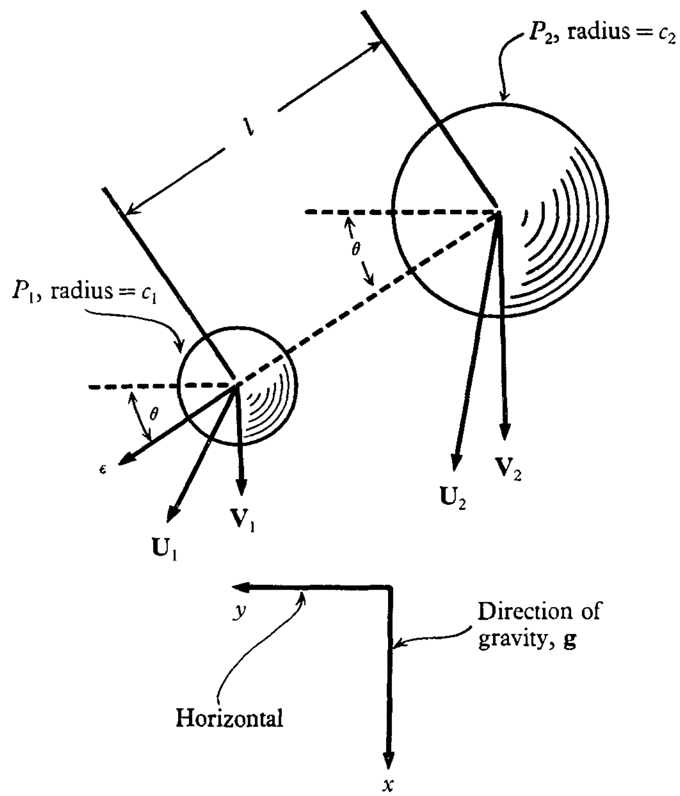

FIGURE 3. Two unequal spheres settling in a viscous fluid.

with a comparable formula for $\mathbf{U}_2$. To express this result in component form, consider the situation depicted in figure 3. If $\mathbf{i}$ and $\mathbf{j}$ denote unit vectors in the x and y directions, respectively, then $\mathbf{V}_1 = \mathbf{i}V_1$, $\mathbf{V}_2 = \mathbf{i}V_2$ and $\boldsymbol{\epsilon} = \mathbf{i}\sin\theta+\mathbf{j}\cos\theta$. In this way we obtain

$$(U_1)_x = V_1+\frac{3}{4}\frac{c_2}{l}V_2(1+\sin^2\theta),$$

$$(U_1)_y = \frac{3}{4}\frac{c_2}{l}V_2\sin\theta\cos\theta,$$

which agree to the first order in c_j/l with corresponding expressions given by Kynch (1959, (3.8)) for spherical particles.

7. Discussion

The analysis leading to (1.1) does not explicitly take account of the possible effects of particle rotation on the force experienced by the particle, P. Rotation of P may arise in either of two ways: first, rotation may occur even in the absence of boundaries, S, in consequence of any initial rotation given P at the start of its trajectory, or by the combined action of gravitational, hydrostatic and hydrodynamic torques acting on a particle released from rest in an infinite fluid (Brenner 1963); secondly, even if the particle has no tendency to rotate in an infinite fluid it may do so in a bounded fluid as a consequence of wall effects. As the equations of motion are linear, the additional forces on P, arising from rotation, may be treated separately from those due to translation and the results superposed.

If P rotates with instantaneous angular velocity $\boldsymbol{\omega}$ as it settles, the boundary condition at its surface is given by $\mathbf{v} = \mathbf{U} + \boldsymbol{\omega} \times \mathbf{r}$, rather than (2.3). Without loss in generality, we let $\mathbf{U}$ be the instantaneous velocity of its 'centre of hydrodynamic stress' C, (Brenner 1963) and measure $\mathbf{r}$ from this point. This choice is not arbitrary but derives from the fact that for combined rotational and translational motions, equation (1.2) is valid only if $\mathbf{U}$ refers to the velocity of point C. Rotation about an axis through C yields no *net* force on the body, at least in an infinite medium, so that the hydrodynamic stresses set up at the surface of a body rotating in this manner produce only a couple.

Consider now the problem of finding the additional rotational field, $(\mathbf{v}_{\text{rot}}, p_{\text{rot}})$, satisfying (2.1), (2.2), (2.4), (2.5) and $\mathbf{v}_{\text{rot}} = \boldsymbol{\omega} \times \mathbf{r}$ on P. The reflexion scheme outlined in (3.1)–(3.7) may be used to solve this problem, except that (3.3) is now replaced by $\mathbf{v}_{\text{rot}}^{(1)} = \boldsymbol{\omega} \times \mathbf{r}$ on P. The initial field, $(\mathbf{v}_{\text{rot}}^{(1)}, p_{\text{rot}}^{(1)})$, corresponds to the rotation of P about C in the *unbounded* fluid. As P experiences no net force due to its rotation in an unbounded fluid (i.e. $\mathbf{F}_{\text{rot}}^{(1)} = \mathbf{0}$), the fluid motion at great distances from it is asymptotically the same as would arise from the action of a 'point couple' equal in strength to the actual couple and situated at C. Thus, (Brenner 1963)

$$\mathbf{v}_{\text{rot}}^{(1)} = -\frac{\mathbf{L}_\infty \times \mathbf{r}}{8\pi\mu r^3} + \mathbf{r}\frac{p_\infty}{2\mu} + o\left(\frac{c}{r}\right)^2, \tag{7.1}$$

where $\mathbf{L}_\infty$ denotes the couple which the body experiences when it rotates about an axis through C with angular velocity $\boldsymbol{\omega}$ in the unbounded fluid, and $p_\infty = p_{\text{rot}}^{(1)}$ denotes the pressure field† arising from this rotation. The latter is a solid spherical harmonic of order -2.

In general, $\mathbf{L}_\infty = O(8\pi\mu c^3\boldsymbol{\omega})$ and $p_\infty = O(4\pi\mu c^3\boldsymbol{\omega}/r^3)$ (see for example, the solutions of Edwardes 1892 and Jeffery 1922 for a rotating ellipsoid), so that

† It was mistakenly assumed in Part 1 that $p_\infty = 0$ for an arbitrary particle. This is true only for an axisymmetric particle rotating about its symmetry axis. Thus, equation (5.9) of Part 1 does not apply to bodies of arbitrary shape as originally stated, but only to axisymmetric bodies rotating about their symmetry axes. Moreover, the boundary shape and the direction of particle rotation relative to this boundary must be such that the streamlines in the bounded fluid lie in circles concentric with the particle axis. That these additional restrictions are *necessary* for the applicability of (5.9) has been confirmed by examining in detail the solution for an ellipsoid rotating about a principal axis, perpendicular to a plane wall.

$\mathbf{v}^{(1)}_{\text{rot}} = O(c/r)^2$. Since $r = O(l)$ on S, we find by arguments similar to those employed in § 3 that the reflexion of $\mathbf{v}^{(1)}_{\text{rot}}$ from S has, at the centre of P, the value $(\mathbf{v}^{(2)}_{\text{rot}})_P = O(c/l)^2$. Hence, from (3.14), the force, $\mathbf{F}^{(3)}_{\text{rot}}$, on P arising from $(\mathbf{v}^{(3)}_{\text{rot}}, p^{(3)}_{\text{rot}})$ is of $O(c/l)^2$. But a term of this order is negligible in our first-order force theory and may be neglected.† It follows that (1.1) (as well as (5.4) and (6.12)) are applicable even in the presence of rotation providing that $\mathbf{U}$ in these relations is interpreted as the velocity of the centre of hydrodynamic stress.

One should not infer from the preceding discussion that rotation, if it occurs, is without significant effect on the translational motion of the particle. A rotating, anisotropic particle falling under the influence of gravity will continuously suffer changes in its instantaneous velocity, $\mathbf{U}$, owing to concomitant changes in the orientation of the particle relative to the direction of the gravity field. For this reason it is desirable to estimate the order-of-magnitude of the angular velocity of a particle which rotates solely in response to wall effects. This is the case most usually encountered in practice. (A homogeneous ellipsoidal particle, for example, has no tendency to rotate in the absence of boundaries.)

The couple on a particle *translating* through a bounded fluid may be estimated from equation (5.7) of Part 1. Though this relation is not quantitatively accurate in the absence of a host of symmetry restrictions, it does furnish the proper orders-of-magnitude in the general case. Thus, in our present notation, the couple is (to dominant terms in c/l)

$$\mathbf{L} = O[8\pi\mu c^3(\nabla \times \mathbf{v}^{(2)}_{\text{transl}})_P].$$

But, from (3.13) we have

$$\mathbf{v}^{(2)}_{\text{transl}} = \boldsymbol{\lambda}\left(\frac{x}{l}, \frac{y}{l}, \frac{z}{l}\right) O\left(\mathbf{U}\frac{c}{l}\right).$$

Taking the curl of this equation and evaluating the resultant expression at the centre of the particle ($x = 0$, $y = 0$, $z = 0$), we find that

$$\mathbf{L} = O(8\pi\mu c^4\mathbf{U}/l^2).$$

But, to dominant terms in c/l, the relationship between the couple on a particle and its angular velocity is of the general form

$$\mathbf{L} = O(8\pi\mu c^3\boldsymbol{\omega}).$$

Upon equating these we obtain‡

$$\omega c/U = O(c/l)^2. \tag{7.2}$$

This estimate is confirmed by the detailed calculations of Wakiya (1959) for the translational motion of an ellipsoid near a vertical, plane wall.

† A slightly more detailed calculation with the aid of (3.16) shows that

$$(F^{(3)}_{\text{rot}}/F^{(3)}_{\text{transl}}) = O[(\omega c/U)(c/l)].$$

If particle rotation occurs solely via wall effects (as is usually the case) then, as shown in the sequel, $\omega c/U$ is at least of $O(c/l)^2$. Thus, the error incurred by neglecting the 'rotational' forces on the particle in this case is even smaller than would otherwise be expected.

‡ Because of their symmetry, spherical particles constitute a degenerate case and produce much smaller effects, as evidenced by the detailed calculations of Faxén (1923) for the translational motion of a sphere parallel to a vertical, plane wall, where it is shown that $\omega c/U = O(c/l)^4$.

Equation (7.2) shows that the relative rate of rotation can normally be expected to be fairly small in the range of c/l values to which (1.1) is applicable.

The range of c/l values for which equation (1.1) and its generalizations are accurate is surprisingly large for a first-order theory. For example, in the case of a spherical particle of radius c falling axially at the centre of a circular cylinder of radius l, equation (1.1) yields

$$\frac{F}{6\pi\mu cU} = \frac{1}{1-k(c/l)},$$

where $k = 2{\cdot}1044$ (Part 1, (3.2)). Values of $F/6\pi\mu cU$ computed from this approximate formula are compared below in table 1 with the 'exact' values given by Haberman & Sayre (1958).

	$F/6\pi\mu cU$	
c/l	Exact	Approximate
0	1·000	1·000
0·1	1·263	1·266
0·2	1·680	1·727
0·3	2·712	2·371

TABLE 1. Stokes-law wall-correction factor for a sphere in a circular cylinder

	$F/6\pi\mu cU$	
h/c	Exact	Approximate
∞	1·0000	1·0000
10·07	0·9308	0·9307
6·132	0·8916	0·8910
3·762	0·836	0·834
2·352	0·768	0·758
1·543	0·702	0·673
1·128	0·660	0·601
1·000	0·645	0·571

TABLE 2. Stokes-law correction factor for two spheres in an unbounded fluid

In a similar vein, for the motion of two equal spheres of radii c moving with equal velocities parallel to their line-of-centres through an otherwise unbounded fluid, equation (6.12) gives, for the force on either sphere,

$$\frac{F}{6\pi\mu cU} = \frac{1-\frac{3}{4}(c/h)}{1-\frac{9}{16}(c/h)^2} = \frac{1}{1+\frac{3}{4}(c/h)},$$

where $2h = l$ is the centre-to-centre distance. Values computed from this approximate formula are compared in table 2 with the exact values of Stimson & Jeffery (1926).

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